History of Key Technologies

Rocket Trajectory Optimization: 1950–1963

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Early Years

I FIRST became interested in the possibility of space travel when, as a child, I read H. G. Wells' novel *First Men on the Moon*. Later, I was presented one Christmas with a copy of Jules Verne's novel *From Earth to Moon and a Trip Around It*, but it was not until I came across the book *Rockets Through Space* by P. E. Cleator, when browsing in a Birmingham library, that I realized that journeys to other worlds need not remain at the level of fantasy. Phil Cleator founded the British Interplanetary Society (BIS) in 1933 and in his book described some experimental firings of liquid-fuel rockets carried out under the auspices of this organization; he speculated that this type of vehicle could be used to explore the solar system. As a young student, I was fascinated by the vistas opened up and immediately joined the society, being rewarded with a set of drawings and commentary describing a solid-fuel rocket thought capable of placing a small party of members on the moon. Unfortunately (or, possibly, fortunately), no sponsor prepared to foot the bill for building this device had stepped forward before the war was declared and the society went into hibernation as I, together with many of its members, joined the armed services.

On being released from the army in 1946, I returned to Cambridge University to complete a mathematics degree course and was there contacted by Len Carter who, as secretary, had been instrumental in amalgamating various rocket groups in the United Kingdom into a reformed BIS. I rejoined the society and began to take an interest in the research work it was encouraging. The calculation of interplanetary rocket trajectories was clearly the problem for which my mathematical training best fitted me to make a contribution. I soon discovered that very little work had been done in this area.

1) The equation of motion of a rocket, namely,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = f + \frac{c}{M}\frac{\mathrm{d}M}{\mathrm{d}t} \tag{1}$$

where v is the rocket velocity, M is its mass, c is its exhaust velocity, and f is the field force per unit mass, had been correctly formulated; upon integration, this yields the equation

$$v_1 - v_0 = \int_{t_0}^{t_1} f\,\mathrm{d}t - c \ln (M_0/M_1) \tag{2}$$

so that for a short burst of motor activity (t_0, t_1), we have

$$\Delta v = v_1 - v_0 = -c \ln R \tag{3}$$

where $R = M_0/M_1$ is the mass ratio for the maneuver. The velocity change Δv is accordingly a convenient measure of the fuel expended and was termed the characteristic velocity for the maneuver (Note: Δv is in the opposite sense to the jet velocity). For a prolonged period of motor activity, during which the velocity change attributable to the field force is not negligible, the characteristic velocity was defined to be $-c \ln R$.

2) The German engineer, W. Hohmann, had studied the problem of transferring a rocket from one circular orbit about the sun into another in the same plane.[1] He had reached the conclusion that transfer along an elliptical orbit, tangential at the two ends of its major axis to the terminal orbits, with short periods of thrust to achieve entry into and exit from the transfer orbit, provided the optimal solution. No proof was offered, but his result is correct provided we limit the number of motor bursts to two. Thirty-five years had to elapse before it was discovered that, in certain circumstances, an improvement could be achieved by permitting more than two periods of motor thrust, but at the cost of increasing the transfer time by a large factor.[2-4] The trick is first to propel the rocket to a great distance from the sun along a cotangential orbit, then to apply a small correcting thrust at aphelion causing the rocket to fall back towards the sun and finally, at perihelion, to enter the target circular orbit via a third thrust. This maneuver takes advantage of the circumstance that changes in orbital elements are most economically generated at a great distance from the sun.

3) The Rumanian engineer, H. Oberth, had published a book on space travel[5] in which he had recognized the mathematical problem of programming the fuel expenditure in a rocket to achieve an optimal result. Thus, assuming the rocket to be subject only to its motor thrust and a gravitational field having potential V, taking the scalar product of both sides of Eq. (1) with v, we obtain the equation

$$v \cdot \frac{\mathrm{d}v}{\mathrm{d}t} = -v \cdot \nabla V + \frac{v \cdot c}{M}\frac{\mathrm{d}M}{\mathrm{d}t} \tag{4}$$

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which is equivalent to

$$\frac{d}{dt}\left(\frac{1}{2}v^2 + V\right) = \frac{\mathbf{v}\cdot\mathbf{c}}{M}\frac{dM}{dt} \tag{5}$$

Denoting the total energy (kinetic and potential) of the rocket per unit mass by E and writing $w = \ln(M_0/M)$ (M_0 = initial mass; w increases with time and is a measure of the propellant expended), this equation may be written

$$\frac{dE}{dw} = -\mathbf{v}\cdot\mathbf{c} \tag{6}$$

Clearly, the energy E increases most rapidly as propellant is expended when the direction of the jet velocity $\mathbf{c}$ is opposite to that of the velocity $\mathbf{v}$ and when the magnitude of $\mathbf{c}$ is as large as possible. Oberth derived this result, but concluded (erroneously) that, for the rocket to achieve maximum energy at the termination of an escape maneuver from the Earth, the rocket thrust should be everywhere tangential to its trajectory. This, I later demonstrated to be untrue.[6] The point to be made is that the effect of varying the thrust direction at any point on the trajectory is not confined to that point and must be calculated at the end of the maneuver. Thus, it may be advantageous to forego a rapid energy increase at the commencement of the maneuver so that the rate of energy increase in the final stages may be as large as possible. It has to be recognized that the analysis of this optimization problem requires the apparatus of the calculus of variations; it cannot be solved, as Oberth supposed, using elementary calculus alone.

Thus, after surveying the available literature, it was clear to me that the primary problem in the field of space rocket trajectories was their optimization with respect to propellant expenditure and that the appropriate analytical tool for tackling this problem was the calculus of variations. Important particular cases were seen to be 1) transfer between elliptical orbits, i.e., a generalization of Hohmann's results; 2) entry into a circular orbit about the Earth allowing for air resistance (Oberth had made a superficial study of this problem and had coined the phrase synergy curve to describe this optimal trajectory); and 3) escape from a terrestrial orbit into the interplanetary transfer orbit.

Initial Attack on the Problem

In my first attempt on the general problem, I decided to take $f = 0$, i.e., no gravitational field or air resistance. Thus, in this null field, it was required to transfer the rocket between two terminals, the rocket velocity at both terminals and the times of departure and arrival being prescribed.

Assuming the motion to take place in the plane of rectangular axes Oxy, if $x(t)$, $y(t)$ are the rocket coordinates at time t and θ is the angle made by the thrust with the x axis, then Eq. (1) shows that

$$\ddot{x} = -\frac{1}{M}\frac{dM}{dt}\cos\theta, \qquad \ddot{y} = -\frac{1}{M}\frac{dM}{dt}\sin\theta \tag{7}$$

It follows that

$$-\frac{d}{dt}(\ln M) = \sqrt{(f\ddot{x}^2 + \ddot{y}^2)} \tag{8}$$

Hence, if t_0 is the time of departure from the terminal (x_0, y_0) and t_1 is the time of arrival at the terminal (x_1, y_1), then

$$\ln R = \int_{t_0}^{t_1}\sqrt{(\ddot{x}^2 + \ddot{y}^2)}\,dt \tag{9}$$

where R is the mass ratio for the maneuver.

We are now presented with a standard calculus of variations problem of minimizing the integral in the last equation by variation of the functions $x(t)$, $y(t)$, subject to the end conditions $x(t_0) = x_0$, $\dot{x}(t_0) = u_0$, $y(t_0) = y_0$, $\dot{y}(t_0) = v_0$, $x(t_1) = x_1$, $\dot{x}(t_1) = u_1$, $y(t_1) = y_1$, $\dot{y}(t_1) = v_1$, where (u_0, v_0) is the departure velocity and (u_1, v_1) is the arrival velocity. Euler's equations should determine the solution and these are found to be

$$\frac{d^2}{dt^2}\left[\frac{\ddot{x}}{\sqrt{\ddot{x}^2+\ddot{y}^2}}\right] = \frac{d^2}{dt^2}\left[\frac{\ddot{y}}{\sqrt{\ddot{x}^2+\ddot{y}^2}}\right] = 0 \tag{10}$$

Referring to the original Eq. (7), it is seen that these are equivalent to the equations

$$\frac{d^2}{dt^2}(\cos\theta) = \frac{d^2}{dt^2}(\sin\theta) = 0 \tag{11}$$

Hence,

$$\cos\theta = A + Bt, \qquad \sin\theta = C + Dt \tag{12}$$

where A, B, C, D, are constants. However, since $\cos^2\theta + \sin^2\theta = 1$, we must take $B = D = 0$ and $A^2 + C^2 = 1$, implying that θ is a constant angle. But, if the rocket thrust direction is constrained to be invariable, it is not possible (in general) to satisfy all eight end conditions and the argument fails to provide a solution.

I was, however, able to recast the problem into another form, when its solution became obvious and the reason for the failure of the standard treatment was revealed[7]: Writing $u = \dot{x}$, $v = \dot{y}$, we have to minimize the integral

$$I = \int_{t_0}^{t_1}\sqrt{(\dot{u}^2 + \dot{v}^2)}\,dt \tag{13}$$

subject to the conditions $u = u_0$, $v = v_0$, at $t = t_0$, $u = u_1$, $v = v_1$, at $t = t_1$, and

$$\int_{t_0}^{t_1} u\,dt = x_1 - x_0, \qquad \int_{t_0}^{t_1} v\,dt = y_1 - y_0 \tag{14}$$

Regarding $u = u(t)$, $v = v(t)$ as parametric equations of a curve C in the uv plane, I represents the length of this curve joining the fixed points $A(u_0,v_0)$ and $B(u_1,v_1)$; this length has to be minimized subject to the conditions of Eq. (14). If s is the arc-length parameter measured along the curve from (u_0,v_0), Eq. (14) can be written

$$\int_C \rho u\,ds = x_1 - x_0, \qquad \int_C \rho v\,ds = y_1 - y_0 \tag{15}$$

where $\rho = dt/ds$. Now $\int_C \rho ds = t_1 - t_0$. Thus, these conditions can be interpreted as requiring that it be possible to distribute matter along the curve C with density ρ such that its mass center should lie at the point P with coordinates $u_P = (x_1 - x_0)/(t_1 - t_0)$, $v_P = (y_1 - y_0)/(t_1 - t_0)$. It is now obvious that the curve C cannot have length less than ($AP + PB$), but that it is possible for C to comprise the straight lines AP, PB, if all of the matter is placed at P. This, then, is the solution to our problem.

Since $\rho = 0$ along AP and PB, the solution requires that the rocket velocity be initially changed from (u_0,v_0) to (u_P,v_P) and then finally from (u_P,v_P) to (u_1,v_1), propellant being expended instantaneously at both terminals. Clearly, this infinite rate of propellant expenditure invalidated our first straightforward approach to the problem.

Primer Vector

In the presence of a gravitational field $(-f, -g)$ per unit mass, the Eq. (1) of rocket motion has components

$$\ddot{x} + f = -\frac{1}{M}\frac{dM}{dt}\cos\theta, \qquad \ddot{y} + g = -\frac{1}{M}\frac{dM}{dt}\sin\theta \tag{16}$$

where we assume f and g to be explicit functions of (x,y,t) and that motion is taking place in the plane of axes Oxy. The mass ratio R for a maneuver is accordingly given by

$$\ln R = \int_{t_0}^{t_1} \sqrt{[(\ddot{x}+f)^2 + (\ddot{y}+g)^2]}\, dt \tag{17}$$

and this has to be minimized subject to the terminal constraints listed in the previous section.

Our preliminary investigation had revealed that, besides having to make allowance for impulsive thrusts, a further difficulty expected to arise was that the Euler characteristic equations associated with the preceding integral would be singular, i.e., the number of integration constants occurring in their solution would be less than their order [vide after Eq. (12)]. To avoid this anomaly, I decided to replace the propellant expenditure integral by

$$\int_{t_0}^{t_1} \sqrt{[(\ddot{x}+f)^2 + (\ddot{y}+g)^2 + \epsilon^2]}\, dt \tag{18}$$

and then let $\epsilon \to 0$. Combined with an allowance for impulsive thrusts, this method was successful in yielding the solution.[8]

The optimal trajectory was shown to comprise a number of null-thrust (free-fall) arcs, separated by junction points at which impulsive thrusts were applied. The conditions determining the junction points were expressed in terms of a primer vector having components (p,q), which were specified along the trajectory by equations

$$\ddot{p} + p\frac{\partial f}{\partial x} + q\frac{\partial g}{\partial x} = \ddot{q} + p\frac{\partial f}{\partial y} + q\frac{\partial g}{\partial y} = 0 \tag{19}$$

The conditions to be satisfied at a junction point were

1) (p,q) must be a unit vector in the direction of the thrust (20a)

2) (p,q) and $(\dot{p},\dot{q})$ must be continuous across the junction (20b)

3) $p^2 + q^2$ must be a maximum (= 1) (20c)

In addition, if the times of departure and arrival are not predetermined, then for minimum propellant expenditure it is necessary that

$$pf + qg + \dot{p}u + \dot{q}v = 0 \tag{21}$$

at both terminals [(u,v) is the rocket velocity].

In the same paper, I calculated the form of the primer vector in an inverse square attraction field, on the assumption that the time of transit was to be chosen optimally. The radial and transverse components of the primer vector were found to be given by

$$p_r = A\cos\theta + Be\sin\theta \tag{22a}$$

$$p_\theta = B(1 + e\cos\theta) - A\sin\theta + \frac{C - A\sin\theta}{1 + e\cos\theta} \tag{22b}$$

e being the eccentricity and θ the true anomaly for the orbit on which the primer is defined (A, B, C, are constants of integration).

The generalization of these results to the case when the transit time is prescribed was made in a later paper[9] and a full consideration of the three-dimensional problem was set out in an article in *Advances in Space Science,* Vol. 1.[10]

Transfer Between Elliptical Orbits

Equations governing the optimal transfer of a rocket between a pair of coplanar elliptical orbits about a center of attraction (γ/r^2) could now be written down by reference to the conditions to be satisfied by the primer vector.[11]

Assuming the number of junction points to be two and taking the polar equation of an orbit in the form $1/r = a + b\cos(\theta + \omega)$, optimal transfer between orbits (a_1,b_1,ω_1), (a_2,b_2,ω_2) was determined by the 11 equations:

$$b_1\cos(\theta_1 + \omega_1) = u_1 - a_1 \tag{23a}$$

$$b_1\sin(\theta_1 + \omega_1) = (u_1 - A_1a_1^{1/2})\tan\phi_1 \tag{23b}$$

$$b\cos(\theta_1 + \omega) = u_1 - a \tag{23c}$$

$$b\sin(\theta_1 + \omega) = (u_1 - A_1a^{1/2})\tan\phi_1 \tag{23d}$$

$$b\cos(\theta_2 + \omega) = u_2 - a \tag{23e}$$

$$b\sin(\theta_2 + \omega) = (u_2 - A_2a^{1/2})\tan\phi_2 \tag{23f}$$

$$b_2\cos(\theta_2 + \omega_2) = u_2 - a_2 \tag{23g}$$

$$b_2\sin(\theta_2 + \omega_2) = (u_2 - A_2a_2^{1/2})\tan\phi_2 \tag{23h}$$

$$\left(\frac{u_1 + a}{A_1a^{1/2}} + 1\right)\cos\phi_1 = \left(\frac{u_2 + a}{A_2a^{1/2}} + 1\right)\cos\phi_2 \tag{23i}$$

$$\left(\frac{u_1}{A_1} - A_1\right)\sin\phi_1 = \left(\frac{u_2}{A_2} - A_2\right)\sin\phi_2 \tag{23j}$$

$$\cos(\theta_1 + \omega - \phi_1) + \frac{a^{1/2}}{A_1}\cos(\theta_1 + \omega)\cos\phi_1$$
$$= \cos(\theta_2 + \omega - \phi_2) + \frac{a^{1/2}}{A_2}\cos(\theta_2 + \omega)\cos\phi_2 \tag{23k}$$

These equations fix the 11 unknowns (a,b,ω) (elements of the transfer orbit), $(1/u_1,\theta_1)$ and $(1/u_2,\theta_2)$ (polar coordinates of the junction points), (ϕ_1,ϕ_2) (angles made by the thrusts with the perpendicular to the radius vector), and constants A_1 and A_2. The characteristic velocity V for the maneuver is found from

$$V = \gamma^{1/2}[u_1(a^{-1/2} - a_1^{-1/2})\sec\phi_1 + u_2(a_2^{-1/2} - a^{-1/2})\sec\phi_2] \tag{24}$$

In general, the solution of Eq. (23) can only be performed numerically and the assistance of a computer is mandatory. Such an instrument was not available to me in 1952, and I accordingly confined my investigations to various special cases, as follows: 1) transfer between concentric circular orbits[12]—the Hohmann solution was verified; 2) transfer between orbits differing only in regard to longitude of perihelion[8,11]; 3) transfer between orbits whose axes are aligned[13,14]; and 4) transfer between orbits of small eccentricity[10]—in this case a rule of attraction of eccentricities was established, according to which the axis of the transfer orbit tends to align itself with the axis of the terminal orbit having the greater eccentricity.

In all cases investigated, it was discovered that transfer via the optimal cotangential ellipse was very nearly as economical as transfer by the overall optimal mode. For a cotangential transfer, the thrusts are both applied along the direction of rocket motion, whereas for the truly optimal mode the thrusts are slightly offset from this direction. The calculation of transfers via cotangential ellipses was studied in the paper listed as Ref. 15.

Alternative derivations of the optimization conditions, Eq. (23), will be found in Refs. 12 and 16.

Arcs of Intermediate Thrust

By expressing the optimal trajectory problem as a problem of Mayer type from the calculus of variations, the necessary conditions calculated earlier were re-established in a more general context.[17,18] In the first paper referred to, it was stated that the only arcs satisfying the optimization conditions were

1) null-thrust arcs and 2) maximum thrust arcs. In correspondence (November, 1960), George Leitmann expressed doubts regarding this conclusion and, in a subsequent investigation[19,20] I was able to demonstrate that a certain class of spiral trajectories in a plane, described under nonmaximum thrust in an inverse square-law field, satisfied all necessary conditions and might, therefore, contribute to an optimal trajectory in such a field of attraction.

A general analysis of these singular solutions of the optimization equations has not hitherto been published and I shall therefore demonstrate their existence immediately.

Along such an intermediate thrust (IT) arc, it has been shown that it is necessary that the primer should have a constant magnitude p, i.e., if $\boldsymbol{p} = (p_1,p_2,p_3)$ denote Cartesian components of the primer vector, then

$$p_i p_i = p^2 \tag{25}$$

where $i = 1, 2, 3$, and the repeated index summation convention is operative throughout this section (thus, $p_i p_i = \Sigma p_i p_i$). If x_i denote the Cartesian coordinates of the rocket at time t and g_i denote the components of the gravitational field (assumed independent of t), the primer must satisfy the differential Eq. (19) along an IT arc and, in our present notation, these can be written

$$\dot{p}_i = q_i \tag{26}$$

$$\dot{q}_i = p_j g_{j,i} \tag{27}$$

where $\boldsymbol{q} = (q_1,q_2,q_3)$ denotes the time derivative of the primer and $g_{j,i} = \partial g_j/\partial x_i$ (in all that follows, an index i following a comma means partial differentiation with respect to x_i). We must also satisfy equations of motion [such as Eq. (16)]; these we express in the form

$$\dot{x}_i = v_i \tag{28}$$

$$\dot{v}_i = g_i + f p_i/p \tag{29}$$

Thus, $\boldsymbol{v} = (v_1,v_2,v_3)$ is the rocket velocity. We have to show that a solution can be found to Eqs. (26–29) satisfying Eq. (25) identically.

Such a solution exists if it is possible, at an initial instant $t = t_0$, to satisfy Eq. (25) and, at the same time, to make the derivatives of $P(t) = p_i p_i$ of all orders vanish. For then, by Taylor's theorem, at any later instant $t > t_0$,

$$\begin{aligned} p_i p_i = P(t) &= P(t_0) + (t-t_0)P'(t_0) + \tfrac{1}{2}!(t-t_0)^2 P''(t_0) + \ldots \\ &= P(t_0) = p^2 \end{aligned} \tag{30}$$

Differentiating $p_i p_i$ with respect to t and equating to zero, we first arrive at the condition

$$2p_i\dot{p}_i = 0 \qquad \text{or} \qquad p_i q_i = 0 \tag{31}$$

Differentiating again and equating to zero, we require that

$$\dot{p}_i q_i + p_i \dot{q}_i = 0 \tag{32}$$

or

$$q_i q_i + p_i p_j g_{j,i} = 0 \tag{33}$$

using Eq. (27).

A third differentiation leads to the requirement

$$2q_i\dot{q}_i + \dot{p}_i p_j g_{j,i} + p_i \dot{p}_j g_{j,i} + p_i p_j g_{j,ik}\dot{x}_k = 0 \tag{34}$$

or

$$p_i q_j (3g_{i,j} + g_{j,i}) + p_i p_j g_{i,jk} v_k = 0 \tag{35}$$

after some exchanges of dummy indices of summation.

A further differentiation will be found to yield the equation

$$\begin{aligned} &(f/p)p_i p_j p_k g_{i,jk} + p_i p_j (3g_{i,k}g_{j,k} + g_{k,i}g_{j,k} + g_k g_{i,jk} + g_{i,jkl}v_k v_l) \\ &+ p_i q_j v_k (4g_{i,jk} + 2g_{j,ik}) + 4q_i q_j g_{i,j} = 0 \end{aligned} \tag{36}$$

We thus have four Eqs. (25), (31), (33), and (35) to be satisfied by 12 unknowns x_i, v_i, p_i, q_i at $t = t_0$, for which there will be ∞^8 possible solutions.[6] Equation (36) will then determine f. Further differentiations of this last equation will provide equations for the time derivatives of f (namely, $\dot{f}, \ddot{f}$, etc.) at $t = t_0$. Using Taylor's theorem, $f(t)$ for $t > t_0$ will then become available.

It has accordingly been shown that initial values of x_i, v_i, p_i, q_i, can be found and f can be defined as a function of t (this determines the rocket thrust), such that the corresponding solution of the differential Eqs. (26–29) satisfies Eq. (25) identically.

Since Eq. (25) is valid for all t, once the conditions of Eqs. (25), (31), (33), and (35) have been satisfied at $t = t_0$, they will remain valid for $t > t_0$ and thus they provide four first integrals of the differential equations. These could be employed to replace four of the differential equations in any numerical integration program, if convenient.

In case the trajectory is to be optimized with respect to the time of transit between the terminals, a further first integral is known to be applicable, namely,

$$p_i g_i - q_i v_i = 0 \tag{37}$$

In this case the number of distinct IT arcs reduces to ∞^7.

If the gravitational field is an inverse square-law attraction to a center, and we confine ourselves to trajectories lying in a plane through the center, then the number of unknowns x_i, v_i, p_i, q_i is eight and the number of distinct IT arcs is ∞^4. Optimization with respect to the time of transit reduces this to ∞^3. However, two of these degrees of freedom correspond to 1) a simple rotation of the trajectory about the center of attraction and 2) a freedom to choose the initial point ($t = t_0$) to be any point on the trajectory. Hence, only an ∞^1 essentially distinct IT arcs exist in this case—this was proved in the papers cited previously,[19,20] where polar equations for the arcs were derived in the form

$$r = a\sin^6\psi/(1 - 3\sin^2\psi), \qquad \theta = -4\psi - 3\cot\psi \tag{38}$$

ψ being the angle between the thrust and the perpendicular to the radius r and a the only remaining parameter open to choice.

It has been shown that this particular spiral arc is nonoptimal, but the families of IT arcs not confined to a plane and in other gravitational fields have not yet, so far as I am aware, been examined for their optimality.

In the case of an inverse square law of attraction ($g = \gamma/r^2$) to a center, Eqs. (26–29) can be expressed in vector form thus:

$$\dot{\boldsymbol{p}} = \boldsymbol{q} \tag{39}$$

$$\dot{\boldsymbol{q}} = \frac{3\gamma}{r^5}(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - \frac{\gamma}{r^3}\boldsymbol{p} \tag{40}$$

$$\dot{\boldsymbol{r}} = \boldsymbol{v} \tag{41}$$

$$\dot{\boldsymbol{v}} = -\frac{\gamma}{r^3}\boldsymbol{r} + f\boldsymbol{p}/p \tag{42}$$

It follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{q}\times\boldsymbol{r} - \boldsymbol{p}\times\boldsymbol{v}) &= \dot{\boldsymbol{q}}\times\boldsymbol{r} + \boldsymbol{q}\times\dot{\boldsymbol{r}} - \dot{\boldsymbol{p}}\times\boldsymbol{v} - \boldsymbol{p}\times\dot{\boldsymbol{v}} \\ &= 0 \end{aligned}$$

by Eqs. (39–42). Hence,

$$q \times r - p \times v = A \tag{43}$$

is a further first integral (taking components, this provides three scalar equations, but only one of these is independent of earlier equations).

Entry into and Escape from Circular Orbits

One of the first space trajectory problems I studied (1951) was the fundamental one of the transference of a rocket from its launching tower into a circular orbit about the Earth. As a first step, I set aside the difficult calculation of the optimal thrust program to penetrate the atmosphere and supposed that the rocket had arrived at a point beyond the atmospheric envelope with a known velocity and that it was required to transfer the rocket from this point into a given coplanar circular orbit using two impulsive thrusts, one applied at the starting point and the other at the orbit, to match the rocket's velocity to the circular velocity at this radius. It was proved[21] that, for minimum propellant expenditure, the transfer ellipse was tangential to the orbit and, therefore, that the final thrust must be in the direction of motion. Characteristic velocities for the maneuver, assuming various values for the initial velocity, were calculated and it was concluded that, for an orbit whose radius exceeded a certain limit, the propellant expenditure was greater than that for complete escape.

Later, the problem of optimal entry into a given circular orbit for a rocket approaching from a great distance was studied.[22] It was assumed that the line of approach to the planet was open to choice. It was then shown that the optimal maneuver is as follows: 1) choose the line of approach so that the point-of-closest approach to the planet is as near to the body's surface as possible (but outside the atmosphere, if any); 2) apply an impulsive thrust at this point in a direction opposite to the motion, to transfer the rocket into an elliptical orbit tangential to the desired circular orbit; and 3) upon arrival at the circular orbit, apply an impulsive thrust to match to the circular velocity. The larger the radius of the circular orbit, the less is the propellant expenditure.

The reverse maneuver of escape from a circular orbit employing impulsive thrusts, was also considered.[23] The case when the thrust magnitude is prescribed (and not infinite) was also researched. This was treated as a calculus-of-variations problem, the object being to calculate the thrust-direction program that yields maximum energy per unit mass for the rocket, in a given gravitational field—a positive final energy indicating that the vehicle will escape from the field. Having elaborated a general theory,[24] this was first applied to the case of a uniform field (to which the attraction along a circular orbit could be approximated, provided the thrust was large so that the period of acceleration was short). It was found that the optimal program required the direction of thrust to be offset from the direction of motion by a small angle in the direction of the field. For a circular orbit just outside the Earth's atmosphere, assuming a constant magnitude of acceleration equal to g, it was found that escape velocity was achieved in 333 s, the throwoff angle for the thrust being initially 3.4 deg, this being steadily reduced to 0 deg as the maneuver proceeded. As expected, comparison with escape by using the same thrust applied tangentially to the trajectory showed that this suboptimal program was only slightly inferior (0.05% less energy) to the optimal one.

In the second part of this paper,[24] the accurate equations for escape from a circular orbit, with constant magnitude of acceleration, were numerically integrated in three cases: 1) thrust directed tangentially to the trajectory; 2) thrust with throwoff angle diminishing linearly with time to zero; and 3) thrust directed transversely, i.e., at right angles to the line joining the rocket to the center of attraction—this program had already been examined by H. S. Tsien.[25] Program 2 proved to be the most economical, the characteristic velocities for programs 1 and 3 being larger by 0.4% and 8.5%, respectively. It was concluded that the tangential thrust program is sufficiently optimal for practical purposes (an unusually strong field of attraction is needed to justify a throwoff angle). The tangential thrust program was examined in detail by D. J. Benney.[26]

A further investigation of the question of optimal escape from a circular orbit, with special consideration of the extreme cases of large and small thrusts, will be found in Ref. 27. An extension of the calculation to the case when the rocket is required to escape with finite velocity at infinity has been given by R. S. Long.[28]

Consideration of Atmospheric Resistance

The most difficult problem in this research area is that of calculating optimal trajectories in the presence of atmospheric resistance. The first problem of this type to be examined was that of programming the propellant expenditure for a vertically ascending sounding rocket so as to achieve maximum height. H. S. Tsien and R. C. Evans solved this problem in 1951.[29] They found that the optimal program separated into three stages: 1) an initial impulsive boost; 2) a variable finite thrust phase; and 3) a coasting phase, with engine shut down, to the maximum height. During the second stage, it is clearly necessary to balance the advantage of large thrust with rapid propellant expenditure against the disadvantage of high velocity with consequent large air resistance. This problem also received attention from A. R. Hibbs[30] and G. Leitmann.[31]

To provide a general theory capable of treating this type of problem where dissipative forces are present, it was necessary to generalize the argument of Ref. 8 to allow for the dependence of the forces applied to the rocket upon the velocity. This extension of the theory was carried through in Ref. 32 and applied to the sounding rocket problem.

Intercontinental Missiles

The military problem of gaining maximum range with a rocket missile is clearly a special case of the general optimization problem we have been considering. Assuming the magnitude of the thrust during the powered phase of the flight to be prescribed, I derived the equations governing the optimal thrust-direction program and applied them to the simple case of motion over an airless flat Earth.[33]

It is proved in my book *Optimal Trajectories for Space Navigation* (Ref. 34, p. 70) that, for a uniform field and zero air resistance, optimization of a rocket trajectory with respect to any cost function demands a bilinear thrust-direction program

$$\tan\theta = \frac{a + bt}{c + dt} \tag{44}$$

where θ is the angle made by the thrust with the horizontal and a, b, c, d, are constants. For the problem we are now considering, $a/c = b/d$ and θ is therefore constant.

In a later paper,[35] the effects of terrestrial rotation and variation of gravity with height were allowed for. A third paper[36] estimated the effect atmospheric resistance and lift forces would have on the optimal trajectory, by treating these forces as small by comparison with the weight force; although this last assumption is unrealistic, some valid qualitative conclusions were reached.

Correctional Maneuvers

In the intellectual climate of the 1950s, there was no shortage of pundits willing to demonstrate, at the drop of a hat, the impracticability of space flight. One problem that some astronomers claimed had been overlooked by astronautical enthusiasts and which they gladly hailed as being unlikely to receive solution was that any small inaccuracy in launching a space vehicle would build up into a very large divergence from the required trajectory at the target planet.[37,38] Having been conditioned to think of extraterrestrial bodies as moving inexorably beyond human control and being imbued with an irrational urge to denigrate proponents of space travel (whom they looked upon as intrusive amateurs), the simple idea that corrections would be applied at points along the rocket trajectory never penetrated their consciousness. It is a sad feature of our human nature that any really new idea or proposal instantly arouses violent antagonism from the establishment who will be affected and who resent any disturbance of their habitual modes of thought; the regular occurrence of this phenomenon through the ages has done nothing to diminish its onset in each generation.

However, although the necessity for correctional maneuvers was obvious and the principles governing them not difficult to elucidate, it seemed judicious to place on record that such calculations had been foreseen. I therefore drew attention[39] to the basic principle that, when a divergence from a computed trajectory has been observed, it is always possible at any convenient later time to make a small velocity correction that will cause the space vehicle to arrive at the next junction point (i.e., point where it is planned to energize the rocket motor) at the appointed time; the thrust to be applied at this junction point will differ slightly from that planned, but the necessary correction is also easily calculated.

The correctional maneuver described in the previous paragraph is nonoptimal for the reason that, if the rocket is found to have been launched into a trajectory deviating from the one originally prescribed, then the planned subsequent junction points are no longer appropriate for an optimal maneuver. Assuming the deviation to be small, R. S. Long and I showed how to calculate the displacement of the next junction point needed to preserve optimal conditions.[40] The consequent saving in propellant expenditure will probably be small in normal circumstances and it seems likely, therefore, that our calculations have never been put to the test.

Since any velocity correction cannot be applied with absolute precision, it is necessary to envisage a sequence of correctional maneuvers and the problem is presented of determining the best timing program for these incidents of control. Considerations of this nature were raised with the author in 1959 when, acting as consultant, he visited the Research Division of Radiation Inc. based at Orlando, Florida. A solution to the problem was found and published in 1960.[41] The result found was that if $t = t_i$ $(i = 1,2,\ldots,n)$ is the sequence of times measured from launching at which corrections are made and $t = T$ is the time of arrival at the target planet, then the quantities $\tau_i = T - t_i$ should form a geometric progression (i.e., $\tau_{i+1}/\tau_i = r$), the common ratio r having a value approximated by $1/e$ $(e = 2.718\ldots)$. The τ_1 is to be taken as large as possible (i.e., the first correction is to be made as soon as possible after launching) and τ_n is to be taken as small as possible (i.e., the final correction is to be left as late as possible). The optimal number n of corrections is determined to be the integer closest to $\ln(\tau_1/\tau_n) + 1$.

Conclusions

I shall terminate this review on a more personal note, which may be of minor historical interest to friends and colleagues who participated in the development of optimization theory during the two decades of the 1950s and 1960s. These are the people from whom I benefitted from meeting and exchanging ideas at the conferences arranged by the International Astronautical Federation and at visits I made to research institutes where they were based. Unhappily, in 1962 my scope for developing personal contacts with those at the center of space research was severely limited by the withdrawal by the U.S. Government of my visitors' visa. At that time, I was located at the University of Canterbury in New Zealand—a very pleasant but somewhat remote post—and I had been invited to spend a sabbatical year working at Lockheed's research unit at Stanford.

I spent a productive sabbatical year at the University of Birmingham in the United Kingdom, during which I assembled all my research material in relation to the optimization problem into the form of a book *Optimal Trajectories for Space Navigation,*[34] which was published the following year. However, my enthusiasm for these investigations declined and upon my return to the United Kingdom (University of Aston) in 1967, I decided to transfer my effort to the field of mathematical physics.

This phase of my life had a happy termination however. The American Institute of Aeronautics and Astronautics was kind enough to make me the first recipient of their Mechanics and Conrol of Flight Award. This is by far the nicest honor to have come my way and I have always been deeply grateful to the Institute for its recognition. Being unable to attend in person to receive the award, my friends Arthur Bryson and John Breakwell stepped into the breach at the ceremony in Huntsville on August 15, 1967, conveyed my appreciation of the honor and best thanks, said many kind words, and thus permitted me to bow out from this stage without too great embarrassment.

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